Analysis of Dynamic Behaviour a Rotating Shaft with Central Mono-Disk

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ABSTRACT

In most moving machineries ranging from micro motors to giant aircraft, rotors are important elements in common. Early investigators noticed the effects of mass imbalance and increasing speeds on the vibrations as the rotors operated near resonance. With present state of the art, speeds in excess of 30,000 rpm should be considered as typical since faster machines would provide better power-to-weight ratio. Doubtlessly, it is crucial to correctly assess the vibration problem here. The present research considers a two dimensional isotropic and flexible horizontal rotor with a symmetrical disk where, amongst others, the gravity and the Coriolis forces are also considered. As the rotor passes from subcritical to supercritical state its dynamic response shows many striking irregularities, reminiscent of chaos as observed in some nonlinear systems.

Keywords: Jeffcott rotor, Phase-Plane Trajectory, Supercritical State, Instability, Chaos.

1. INTRODUCTION

Rotors are principal elements in all moving machineries which constitute the most common mechanical systems. Examples include machine tools, industrial turbo-machinery, aircraft gas turbine engines and turbo pumps. Vibrations caused by mass imbalance are a common problem in rotating machinery. Imbalance occurs if the mass centre of the rotor does not coincide with its axis of rotation. Even though higher speeds cause much greater centrifugal imbalance forces the current trend of higher power density invariably leads to higher operational speeds in rotating machinery. For example, speeds as high as 30,000 rpm are typical in current high-speed machining applications.

Early investigators noticed the occurrence of excessive vibrations of rotors when the speed of rotation came close to the natural frequency. This had been termed as the “critical speed”. Jeffcott¹ in 1919 considered the lateral vibrations of a flexible shaft in the vicinity of critical speeds. In his analytical model, which consisted of a single disk centrally mounted on an isotropic simply-supported shaft, the disk did not wobble. Therefore, the angular velocity vector and the angular momentum vector were collinear and no gyroscopic moments were present. Researchers like De Laval and Föppl showed that operation beyond the critical speed is a possibility where a significant reduction of vibrations can be expected. Studies on Jeffcott rotors have been reported by Hussein et. al², who investigated the whirl phenomenon at constant speed under varying stiffness and damping combinations. Pavlovskai et al³ considered a two degree-of-freedom model of the Jeffcott rotor with a snubber ring subjected to out of
balance excitation where a non-linear piece-wise dynamical system was formed showing bifurcations in the response. Karpenko et al.\(^4\) carried out bifurcation analysis of a preloaded Jeffcott rotor who report the influence of preload and damping on the system dynamics.

Despite greatest care, rotors cannot be fully balanced dynamically and at high speeds, operative centrifugal actions tend to intensify vibrations. With nonlinear flexible systems sub-harmonic vibrations may occur at periods that are integer multiples of the fundamental\(^{19}\).

An accurate prediction of the dynamic characteristics is vital to the designer of rotating machinery. Most rotors are axisymmetric and their analysis is somewhat simpler. Some rotors, however, do not possess this symmetry with the result that much complication is introduced in their analysis. Further, rotors formed with flexible shafts with large disks, set up gyroscopic and rotatory inertia couples that introduce new and complex phenomena.

For general rotors, the question of critical speeds of continuous rotor systems have been addressed by Dimentberg\(^5\). The frequency equations and critical speeds of a straight circular rotor were obtained by Eshleman and Eubanks\(^7\) who included the effects of transverse shear, rotatory inertia and gyroscopic moments. Finite element methods were used to determine critical speeds of straight circular rotors by Nelson et. al.\(^{8, 9}\). Rouch and Rao\(^{10}\) developed the stiffness, mass and gyroscopic matrices of a rotating beam element with inclusion of the shear deflection term. Gmur and Rodrigues\(^{11}\) studied the dynamics of tapered circular rotors through finite element modelling. The effect of shear deflection and rotatory inertia on the critical speeds of the rotor was taken into account by R. Grybos, Gliwice\(^6\), which is of interest especially when a critical speed of higher order is concerned and the ratio of slenderness of a rotor is small.

Ozguven and Ozkan\(^{12}\) presented the combined effects of shear deformation and internal damping to analyse the natural whirl speeds and unbalance response of homogeneous rotor-bearing systems.

Information about the stability of vibratory motions becomes essential for ensuring better designs of rotor-bearing systems and operational safety. The effects of bearing and shaft asymmetries on the stability of the rotor has been reported by Ganesan\(^{13}\). Gunter Jr. and Trumpler\(^{14}\) evaluated the stability of the single disk rotor with internal friction on damped, anisotropic supports. Wettergren and Olsson\(^{15}\) considered a horizontal rotor with a flexible shaft supported in flexible bearings and found that major instabilities appear near the imbalance resonance and remarked that the resonances due to gravity near one half of the major critical could be reduced with enhanced material damping. Hull\(^{16}\) experimentally scrutinized the whirling of a rotor in anisotropic bearings and also studied the backward whirling process theoretically. Smith\(^{17}\), while studying the motion of an asymmetric rotor in flexible anisotropic bearings, found that the motion was marked by unstable ranges bounded by critical speeds with instabilities at speeds lower than the principal critical. Rajalingham et al.\(^{18}\) considered the influence of external damping on the stability and dynamic response of single disk horizontal rotors with anisotropic bending stiffness characteristics. They showed that sufficient damping can suppress instability.
2. NOMENCLATURE

- $a_1, a_2$: Lateral vibration amplitude (m)
- $c$: Internal damping co-efficient (N-s/m)
- $C_1, C_2$: Amplitude terms in equations (13) and (14) (m)
- $g$: Gravitational acceleration (m/s²)
- $k$: Bending stiffness of the shaft at disk location (N/m)
- $L$: Span of the simply-supported rotor shaft (m)
- $m$: Equivalent mass of the rotor system (kg)
- $p$: Natural frequency of the non-rotating undamped shaft-disk system (rad/s)
- $r$: Lateral deflection of the shaft (m)
- $t$: time (s)
- $x, y$: rotating frame of reference (coordinate system)
- $x, y_1$: stationary frame of reference (coordinate system)
- $\dot{x}, \dot{y}$: Velocity components along rotating frame of reference (m/s)
- $\ddot{x}, \ddot{y}$: Acceleration components along rotating frame of reference (m/s²)
- $\theta$: Angular displacement at any instant of time ‘t’ (rad)
- $\omega = \dot{\theta}$: Angular velocity at any instant of time ‘t’ (rad/s)
- $\lambda$: Amplitude of deflection (m)
- $\psi$: Slope at disk location (rad)
- $\zeta_1$: Damping ratio (internal or material damping)
- $\Omega$: Vibration frequency of the rotating shaft (rad/s)
- $\omega_1, \omega_2$: Vibration frequencies of the rotating shaft (rad/s)
- $\phi_1, \phi_2$: Phase angle terms in equations (13) and (14) (rad)

3. MATHEMATICAL ANALYSIS

To build a mathematical model, we start with a simple prototype of a rotor consisting of a single disk mounted at the mid span of a flexible circular simply supported shaft. This is the Jeffcott rotor, first reported by Jeffcott(1) in 1919. Though simple, it exemplifies many industrial applications. The flexibility of the shaft is considered to be high such that those of the bearings can be neglected. The schematics of the Jeffcott rotor are shown in fig.1.

![Fig.1: Jeffcott Rotor Schematics](image_url)

The dynamic deflection curve of the shaft around its fundamental vibration mode can be approximated by a half-cycle sinusoid given by:

$$\text{Deflection} = r = \lambda \sin \left( \frac{\pi z}{L} \right) \quad \text{and} \quad \text{slope} = \psi = \frac{dr}{dz} = \frac{\pi}{L} \lambda \cos \left( \frac{\pi z}{L} \right)$$

At the mid span, $z = L/2$, and $\psi \big|_{z=L/2} = 0$ and $r \big|_{z=L/2} = \lambda \cdot$
Thus the slope at the disk location remains zero within the fundamental vibration mode and even somewhat beyond. We now make the following assumptions:

1. The shaft is thin and flexible and its mass can be neglected.
2. The disk is balanced (i.e., its mass centre G coincides with the shaft centre line), and since the slope $\varphi = 0$ at the disk location, it whirls cylindrically without wobbling. No gyroscopic effect is, therefore, considered.
3. The shaft shows linear elasticity and the end bearings are considered as rigid simple supports.

Figure 2 shows a balanced Jeffcott rotor where the position of the centre of rotation is ‘O’ (shaft centre) and the instantaneous position of the centre of mass of the disk is ‘G’. The co-ordinate system x-y is assumed to rotate with a uniform angular velocity $\omega$ equal to the speed of the disk. This is shown in fig.3. As the shaft rotates, the centre of mass ‘G’ of the disk revolves about ‘O’ with angular velocity $\omega = \dot{\theta}$ and the length of the radius vector OG (= r), in general, changes. Kinematically\(^{(23)}\), the point ‘G’ experiences four kinds of accelerations, namely, (1) the centripetal acceleration $r\dot{\theta}^2$ directed along GO, (2) the radial acceleration $\ddot{r}$ directed along OG, (3) the tangential acceleration $r\dot{\theta}$ perpendicular to OG, and (4) the Coriolis acceleration $2\dot{r}\dot{\theta}$ perpendicular to OG, with angular velocity $\dot{\theta} = \omega =$ constant, the angular acceleration $\ddot{\theta} = 0$, and consequently, the tangential acceleration vanishes. We can now visualize the dynamics of the rotating shaft-disk system.

The forces that are active may be considered in two categories as follows:

1. External, the example of which is the gravity force ‘mg’ (fig.8).
2. Forces arising out of motion, such as the inertia force, elastic force, centrifugal force, Coriolis force and damping force. These are shown in fig.4 to 7 and fig.9, respectively.
4. EQUATIONS OF MOTIONS

Figure 4 to 9 show projections of different forces acting on the system resolved along x-direction and y-direction respectively. These are:

- Inertia force components: \(-m\ddot{x}\) and \(-m\ddot{y}\) (fig.4)
- Coriolis force components: \(-2m\omega\dot{y}\) and \(2m\omega\dot{x}\) (fig.5)
- Centrifugal force components: \(m\omega^2x\) and \(m\omega^2y\) (fig.6)
- Elastic force components: \(-kx\) and \(-ky\) (fig.7)
- Gravity force components: \(-mg\sin\theta\) and \(-mg\cos\theta\) (fig.8)
- Damping force components: \(-c\dot{x}\) and \(-c\dot{y}\) (fig.9)

The resulting equations of motion are:

\[ M \ddot{q} + C \dot{q} + (K+N)q = f \]  \hspace{1cm} (1)

where:

- \(M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\) is the symmetric mass matrix,
- \(C = \begin{bmatrix} c & 2m\omega \\ -2m\omega & c \end{bmatrix}\) is the damping and Coriolis matrix,
- \(K = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\) is the symmetric stiffness matrix,
- \(N = \begin{bmatrix} -m\omega^2 & 0 \\ 0 & -m\omega^2 \end{bmatrix}\) is the centrifugal action matrix.

\(q = q(t)\) is the generalized coordinates vector of the rotor and \(f = f(t)\) is a forcing function vector.
For a vertical shaft, $f(t) = 0$, equation (1) can be reduced to the following set:

$$\ddot{x} + (p^2 - \omega^2) x + 2 \alpha \dot{y} + 2 \zeta_x p \dot{x} = 0 \quad \text{------------------------ (2)}$$

$$\ddot{y} + (p^2 - \omega^2) y - 2 \alpha \dot{x} + 2 \zeta_y p \dot{y} = 0 \quad \text{------------------------ (3)}$$

Where $\zeta = \frac{c}{2p} = \text{the damping ratio of the system}$, $p = \sqrt{\frac{k}{m}} = \text{the natural frequency of the non-rotating shaft-disk system}$.

Equations (2) and (3) are coupled second order homogeneous differential equations. When material damping is the principal dissipative agent, we may logically neglect the same (being < 1% of critical). Then equations (2) and (3) can be further simplified to:

$$\ddot{x} + (p^2 - \omega^2) x + 2 \alpha \dot{y} = 0 \quad \text{------------------------ (4)}$$

$$\ddot{y} + (p^2 - \omega^2) y - 2 \alpha \dot{x} = 0 \quad \text{------------------------ (5)}$$

We may assume for $\omega \neq p$, the solutions as: $x = a_1 \sin(\Omega t + \alpha)$ and $y = a_2 \cos(\Omega t + \alpha)$.

Substituting these in eqns. (4) and (5), we obtain,

$$\left( p^2 - \omega^2 - \Omega^2 \right) a_1 - 2 \alpha \Omega a_2 = 0 \quad \text{------------------------ (6)}$$

$$- 2 \alpha \Omega a_1 + \left( p^2 - \omega^2 - \Omega^2 \right) a_2 = 0 \quad \text{------------------------ (7)}$$

This homogeneous system of algebraic equations has non-trivial roots only if the determinant of the coefficients of the system is zero.

$$\left| \begin{array}{cc}
(p^2 - \omega^2 - \Omega^2) & - 2 \alpha \Omega \\
- 2 \alpha \Omega & (p^2 - \omega^2 - \Omega^2)
\end{array} \right| = 0 \quad \text{------------------------ (8)}$$

Whence, the two roots of $\Omega^2$ are: $(p + \omega)^2$ and $(p - \omega)^2$.

The general solutions for equations (9) and (10) can then be written as:

$$x = C_1 \sin (\omega t + \phi_1) + C_2 \sin (\omega t + \phi_2) \quad \text{------------------------ (9)}$$

$$y = C_1 \cos (\omega t + \phi_1) - C_2 \cos (\omega t + \phi_2) \quad \text{------------------------ (10)}$$

Where $\omega_1 = p + \omega$ and $\omega_2 = p - \omega$. The constants $C_1, C_2, \phi_1$ and $\phi_2$ are determined by the initial conditions of motion.

For both $\omega \leq p$ and $\omega > p$, any disturbance leads to vibrations of constant amplitudes. The two frequencies of vibrations $\omega_1$ and $\omega_2$, as observed in the rotating co-ordinate system, differ from the natural frequency $p$ of the stationary shaft by $\omega$.

For a horizontal shaft, $f(t) = \left[ -mg \sin \theta \right]$. Assuming a particular solution of the form:

$x = a_1 \sin \omega t$ and $y = a_2 \cos \omega t$, and substituting them in equations (4) and (5), we arrive at a non-homogeneous system of algebraic equation for the amplitudes $a_1$ and $a_2$ as

$$-a_1 \omega^2 + (p^2 - \omega^2) a_1 - 2a_2 \omega^2 = -g \quad \text{------------------------ (11)}$$
\[-a_2\omega^2 + \left(p^2 - 4\omega^2\right)a_2 - 2a_1\omega^2 = -g \quad \text{(12)}\]

Solving the above equations, we obtain
\[a_1 = a_2 = -\frac{g}{l(p^2 - 4\omega^2)}\]

The denominators of the amplitude terms \(a_1\) and \(a_2\) in the above equations are common and both of them \(\rightarrow \infty\) when \(\omega = p/2\). This defines an instability.

The complete solution for the gravity excited undamped vibratory motion (equations (11) and (12)) can be represented as:

\[x = C_1\sin(\omega t + \phi_1) + C_2\sin(\omega_2 t + \phi_2) - \frac{g}{\left(p^2 - 4\omega^2\right)}\sin \omega t \quad \text{(13)}\]

\[y = C_1\cos(\omega t + \phi_1) - C_2\cos(\omega_2 t + \phi_2) - \frac{g}{\left(p^2 - 4\omega^2\right)}\cos \omega t \quad \text{(14)}\]

Examination of equations (13) and (14) reveals a vibratory motion of three frequencies, namely \(\omega_1\), \(\omega_2\) and \(\omega\). When damping is considered, the first two terms die down with time and a steady state is expressed by the third term. For small material damping, however, this decay time will be appreciably long.

5. **NUMERICAL EXAMPLE AND DIGITAL SIMULATION**

Fig. 10 shows a proposed horizontal Jeffcott Rotor supported on ball bearings (permitting small end slopes) with a central balanced circular disk. The rotor is driven by a motor directly through a flexible coupling. The two bearing pedestals are adjustable vertically so as to keep the shaft horizontal when stationary.

![Fig 10: The Proposed Jeffcott Rotor](image)

\[m = \text{Equivalent translatory mass (mass of steel disk} + 49\% \text{ of the mass of steel shaft)} = 0.628 \text{ kg}; \text{ } k = \text{stiffness of shaft} = 2301 \text{ N/m}.\]

Defining the state vectors: \(v_x = \dot{x}, v_y = \dot{y}\), and noting that \(\dot{v}_x = \ddot{x}, \dot{v}_y = \ddot{y}\), equations (11) and (12) can be readily reduced to the following set of four first order ordinary differential equations as follows:
\[
\begin{align*}
\begin{bmatrix}
\dot{v}_x \\
\dot{x} \\
\dot{v}_y \\
\dot{y}
\end{bmatrix} &=
\begin{bmatrix}
-2\zeta_1 p & -(p^2 - \omega^2) & -2\omega & 0 \\
1 & 0 & 0 & 0 \\
2\omega & 0 & -2\zeta_1 p & -(p^2 - \omega^2) \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
v_x \\
x \\
v_y \\
y
\end{bmatrix} +
\begin{bmatrix}
-\sin \omega t \\
0 \\
-\cos \omega t \\
0
\end{bmatrix} g 
\end{align*}
\]

The equations set ((15) to (18)) can be conveniently integrated by Runge-Kutta algorithms\(^{(22)}\).

6. **RESULTS AND DISCUSSION**

In the following depictions we represent cases of subcritical (at \(\omega/p = 1/4\)) to supercritical (at \(\omega/p = 4/3\)) rotors at damping ratios \(\zeta_i\) in the practical range (i.e., 0.5 % to 1 %, typical for material damping).

![Fig. 11(a): Phase-Plane Plot](image1)

![Fig. 11(b): Displacement-Time Plot](image2)

![Fig. 12(a): Phase-Plane Plot](image3)

![Fig. 12(b): Displacement-Time Plot](image4)

Fig. 11(a) shows the phase plane plot for the shaft centre along the rotating x-axis with the frequency ratio \(\omega/p = 1/4\) and damping ratio \(\zeta_1=0.005\). The two oval lobes on the RH and the LH sides of the \(x = 0\) axis are nearly identical. For low values of time \(t\), the plots evolve in a very interesting fashion: starting at the origin of the phase-plane axes (as governed by the chosen initial conditions), the RH lobe is incompletely described and a small oval kink is generated near the origin whence the trajectory breaks off and begins to describe the LH lobe and before completing the same, another kink forms...
near the origin and the trajectory repeats its course on the RH side. However, at very high values time ‘t’, the phase-plane trajectory becomes simpler in appearance as it assumes a near elliptic shape (not shown in the plot) about the origin but never repeating the same path. This happens because with time the small amplitude transients at frequencies $\omega_1 = (p + \omega)$ and $\omega_2 = (p - \omega)$ die down and large amplitude oscillations at $\omega$ predominate. Examination of the displacement-time plot in fig.11(b) reveals the multiple frequency character of the same.

Fig. 12(a) and 12(b) show very slight differences from fig. 11(a) and 11(b) as the damping ratio $\zeta_1$, though doubled, remains small. The displacement-time plot, however, indicates a faster decay of the transients.

Fig.13(a) shows the phase-plane plot for the frequency ratio $\omega/p = 1/2$ at 1 % material damping. This is a clear case of instability as the trajectory spirals outwards. Fig.13(b), representing the displacement-time plot, also diverges with oscillations having predominantly a single frequency of $\omega = p/2$ equaling the driving frequency. Note that in this case, $\omega_2 = p/2$ and $\omega_1 = 3p/2$, but the latter frequency term is not noticeable.
Fig. 14(a) and 15(a) show the phase-plane plots at a frequency ratio \( \omega/p = \frac{3}{4} \) but at two damping ratios, 0.005 and 0.01, respectively. These plots have fascinating shapes where the trajectory begins nearly at the origin and after describing part of a smaller loop in the RH side, the plot swings to complete a larger loop covering the LH side and then back in the RH side to partly finish a smaller loop there. Thence, it begins with the creation of a smaller loop in the LH side and process keeps on alternating between the two sides. Notably, the trajectories never seemed to follow the same path in the range of computation. The displacement-time plots, fig.14(b) and 15(b), nearly identical in appearance, show presence of multiple frequencies \( \omega_1 = \frac{7p}{4}, \omega_2 = \frac{p}{4} \) and \( \omega = \frac{3p}{4} \) with the forcing term \( \omega \) predominating but the effect of damping is not significant.

Fig.16(a, b) depict the response at \( \omega/p = 1 \) (the critical state) where an instability is not found. By equations (13) and (14), the amplitude of the steady state response is given by \( g/3p^2 \), which is finite. The three frequencies are: \( \omega_1 = 2p, \omega_2 = 0 \) and \( \omega = p \), and in fig.16(b), bi-harmonic oscillations at frequencies \( p \) and \( 2p \) are observable initially. After about 1.5 s, vibrations at frequency \( 2p \) die down due to damping and harmonic motion at frequency \( \omega \) is seen to persist. The phase-plane plot shown in fig.16(a), corroborates this showing a nearly circular limit cycle at \( t > 1.5 \) s.
Fig. 17(a) and 18(a) depict supercritical response at a frequency ratio of $\omega/p = 4/3$ and at two damping ratios. The phase-plane trajectories are composed of three loops, dispersed more or less symmetrically about the $x = 0$ line. Here again, none of the paths are traversed more than once. Clearly, three frequency terms ($\omega_1 = 141.2$ rad/s, $\omega_2 = 20.2$ rad/s and $\omega = 60.5$ rad/s with respective periods of 44.5 ms, 311 ms and 103.8 ms) are notable. At the lower damping the displacement cycles do not repeat at intervals of 311 ms. At higher damping, however, the cycles seem to be more or less repetitive. This shows that higher damping is effective in the supercritical region.

In fig. 11 to 18, the x-axis has been chosen for reference. If the y-axis be chosen, like results would be obtained. Further, the system is seen to become unstable beyond $\omega/p = 3/2$.

7. CONCLUSION

The phase-plane trajectories show a striking peculiarity in common: they never form a closed loop in the range of computation (1800 ms) considered by the authors. Also, none of the paths are traversed more than once and no limit cycling is observed in the phase plane trajectories excepting fig. 16(a, b). Further, these patterns show a marked change when the initial conditions are slightly altered. The irregularities in the response are reminiscent of chaotic behaviour which, for linear system, cannot be expected. For horizontal rotors, it is always dangerous to operate the system at $\omega = p/2$ and $\omega > 3p/2$. 

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REFERENCES


